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Calculation of the average Green's function of electrons in a stochastic medium via higher-dimensional bosonization

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Abstract. The disorder-averaged single-particle Green's function of electrons subject to a time-dependent random potential with long-range spatial correlations is calculated by means of bosonization in arbitrary dimensions. For static disorder our method is equivalent to conventional perturbation theory based on the lowest-order Born approximation. For dynamic disorder, however, we obtain a new non-perturbative expression for the average Green's function. Bosonization also provides a solid microscopic basis for the description of the quantum dynamics of an interacting many-body system via an effective stochastic model with Gaussian probability distribution.

Over the past 20 years the motion of a quantum particle in a time-dependent random potential has been the subject of many works [1–3]. Most authors have focused on the case of a single isolated electron, where numerical as well as non-perturbative analytical methods are available [1]. The equally important problem of calculating the average single-particle Green's function in the presence of a filled Fermi sea has not received much attention [2]. The purpose of the present paper is to show that the functional integral formulation of bosonization [4–8] offers a new non-perturbative approach to this problem in arbitrary dimensions d.

The bosonization approach in $d > 1$ has recently been re-discovered by Haldane [9], who generalized the earlier ideas of Luther [10]. For a detailed description of the geometric construction leading to higher-dimensional bosonization we would like to refer the reader to [9]. Here we briefly recall the basic features of Haldane's construction. The first step is the subdivision of the Fermi surface into disjoint patches $\tilde{K}^{\alpha}_{\Lambda}$ and the introduction of a collection of local coordinate systems with origins on the Fermi surface (this is called an *atlas* [11]). In d dimensions each patch covers an area Λ^{d-1} of the Fermi surface, where the cut-off Λ should be chosen such that within a given patch the curvature of the Fermi surface can be locally neglected. For spherical Fermi surfaces, this means that Λ should be chosen small compared with the Fermi wavevector k_F. Each patch is then extended into a d-dimensional box $K^{\alpha}_{\Lambda,\lambda}$ with radial height λ and volume $\Lambda^{d-1}\lambda$. One proceeds by defining local density operators associated with the boxes, $\hat{\rho}^{\alpha}_{\boldsymbol{q}} = \sum_{\boldsymbol{k}} \Theta^{\alpha}(\boldsymbol{k})\hat{c}^{\dagger}_{\boldsymbol{k}}\hat{c}_{\boldsymbol{k}+\boldsymbol{q}}$, where $\hat{c}_{\boldsymbol{k}}$ is the annihilation operator of an electron with wavevector $\boldsymbol{k}$ (for simplicity we consider spinless electrons in this work), and the cut-off function $\Theta^{\alpha}(\boldsymbol{k})$ is unity for $\boldsymbol{k} \in K^{\alpha}_{\Lambda,\lambda}$ and vanishes otherwise. In the functional bosonization approach [4–8] the calculation of the interacting Green's function is then mapped via a Hubbard–Stratonovich transformation onto the problem of calculating the *average* Green's function of an effective non-interacting system in a dynamic random potential ϕ^{α}, which couples to the local densities $\hat{\rho}^{\alpha}$.

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To see the connection between functional bosonization and random systems more clearly, let us briefly describe the basic features of the functional bosonization approach developed in [6–8]. For simplicity, let us consider spinless fermions. The starting point is the imaginary time Grassmannian functional integral representation of the Matsubara Green's function of an interacting many-body system with two-body density–density interactions $f_{\boldsymbol{q}}^{\alpha\alpha'}$

$$G(k) \equiv G(\boldsymbol{k}, \mathrm{i}\tilde{\omega}_n) = -\beta \frac{\int \mathcal{D}\{\psi\}\, e^{-S_{mat}\{\psi\}} \psi_k \psi_k^\dagger}{\int \mathcal{D}\{\psi\}\, e^{-S_{mat}\{\psi\}}} \tag{1}$$

where the Euclidean action $S_{mat}\{\psi\}$ is the following functional of the Grassmann field ψ

$$S_{mat}\{\psi\} = S_0\{\psi\} + S_{int}\{\psi\} \tag{2}$$

$$S_0\{\psi\} = \beta \sum_k \left[-\mathrm{i}\tilde{\omega}_n + \epsilon_{\boldsymbol{k}} - \mu\right] \psi_k^\dagger \psi_k \tag{3}$$

$$S_{int}\{\psi\} = \frac{\beta}{2V} \sum_q \sum_{\alpha\alpha'} f_{\boldsymbol{q}}^{\alpha\alpha'} \rho_{-q}^{\alpha} \rho_q^{\alpha'}. \tag{4}$$

Here $\epsilon_{\boldsymbol{k}}$ is the energy dispersion, μ is the chemical potential, β is the inverse temperature, and V is the volume of the system. The local density operator is now represented by a composite Grassmann field $\rho_q^\alpha = \sum_k \Theta^\alpha(\boldsymbol{k}) \psi_k^\dagger \psi_{k+q}$. Throughout this work we shall use the convention that $k = [\boldsymbol{k}, \mathrm{i}\tilde{\omega}_n]$ and $q = [\boldsymbol{q}, \mathrm{i}\omega_m]$, where the fermionic frequencies are $\tilde{\omega}_n = 2\pi(n + \frac{1}{2})/\beta$ and the bosonic ones are $\omega_m = 2\pi m/\beta$. Introducing bosonic auxiliary fields ϕ^α via a Hubbard–Stratonovich transformation, $G(k)$ can be exactly rewritten as

$$G(k) = \int \mathcal{D}\{\phi^\alpha\} \mathcal{P}\{\phi^\alpha\} [\hat{G}]_{kk} \equiv \left\langle [\hat{G}]_{kk} \right\rangle_{S_{eff}}. \tag{5}$$

Here $\hat{G}^{-1}$ is an infinite matrix in momentum and frequency space, with matrix elements given by the formal Dyson equation $[\hat{G}^{-1}]_{kk'} = [\hat{G}_0^{-1}]_{kk'} - [\hat{V}]_{kk'}$, where $\hat{G}_0$ is the non-interacting Matsubara Green's function matrix, $[\hat{G}_0]_{kk'} = \delta_{kk'} G_0(k)$, with $G_0(k) = [\mathrm{i}\tilde{\omega}_n - \epsilon_{\boldsymbol{k}} + \mu]^{-1}$. The generalized self-energy matrix $\hat{V}$ is $[\hat{V}]_{kk'} = \sum_\alpha \Theta^\alpha(\boldsymbol{k}) V_{k-k'}^\alpha$, with $V_q^\alpha = \frac{\mathrm{i}}{\beta} \phi_q^\alpha$. The normalized probability distribution $\mathcal{P}\{\phi^\alpha\}$ is

$$\mathcal{P}\{\phi^\alpha\} = \frac{\mathrm{e}^{-S_{eff}\{\phi^\alpha\}}}{\int \mathcal{D}\{\phi^\alpha\}\, \mathrm{e}^{-S_{eff}\{\phi^\alpha\}}} \tag{6}$$

where the effective action for the ϕ^α field is of the form $S_{eff}\{\phi^\alpha\} = S_2\{\phi^\alpha\} + S_{kin}\{\phi^\alpha\}$, with

$$S_2\{\phi^\alpha\} = \frac{1}{2} \sum_q \sum_{\alpha\alpha'} [\underline{\tilde{f}}_{-q}^{-1}]^{\alpha\alpha'} \phi_{-q}^\alpha \phi_q^{\alpha'} \tag{7}$$

$$S_{kin}\{\phi^\alpha\} = -\mathrm{Tr}\, \ln[1 - \hat{G}_0 \hat{V}]. \tag{8}$$

Here $\underline{\tilde{f}}_q$ is a matrix in the patch indices, with matrix elements $[\underline{\tilde{f}}_q]^{\alpha\alpha'} = \frac{\beta}{V} f_{\boldsymbol{q}}^{\alpha\alpha'}$. The action $S_{kin}\{\phi^\alpha\}$ can be calculated perturbatively by expanding equation (8) in powers of the ϕ^α field. The validity of this expansion is controlled by the generalized closed-loop theorem, which is discussed in detail in [7]. At the level of the Gaussian approximation one obtains

$$S_{kin}\{\phi^\alpha\} \approx \frac{V}{2\beta} \sum_q \sum_{\alpha\alpha'} \Pi_0^{\alpha\alpha'}(q) \phi_{-q}^\alpha \phi_q^{\alpha'} \tag{9}$$

where the 'patch' polarization is given by

$$\Pi_0^{\alpha\alpha'}(q) = -\frac{1}{2\beta V}\sum_k \Big[\Theta^{\alpha}(\boldsymbol{k})\Theta^{\alpha'}(\boldsymbol{k}+\boldsymbol{q})G_0(k)G_0(k+q) + \Theta^{\alpha'}(\boldsymbol{k})\Theta^{\alpha}(\boldsymbol{k}-\boldsymbol{q})G_0(k)G_0(k-q)\Big]. \tag{10}$$

To calculate the matrix elements $[\hat{G}]_{kk}$ it is convenient to choose the patches larger than the range of the interaction in momentum space. In this case we may write [8] $[\hat{G}]_{kk} = \sum_\alpha \Theta^\alpha(\boldsymbol{k})[\hat{G}^\alpha]_{kk}$. It is convenient to shift the origin in momentum space to the centre $\boldsymbol{k}^\alpha$ of patch $\tilde{K}^\alpha_\Lambda$ on the Fermi surface: $\boldsymbol{k} = \boldsymbol{k}^\alpha + \boldsymbol{q}$, $\boldsymbol{k}' = \boldsymbol{k}^\alpha + \boldsymbol{q}'$. For small $\boldsymbol{q}$ we may then linearize the energy dispersion, $\epsilon_{\boldsymbol{k}^\alpha+\boldsymbol{q}} - \mu \approx \boldsymbol{v}^\alpha \cdot \boldsymbol{q}$. It follows that the infinite matrix $\hat{G}^\alpha$ satisfies

$$\sum_{\tilde{q}'}\Big[\delta_{\tilde{q},\tilde{q}'}[G_0^\alpha(\tilde{q})]^{-1} - V^\alpha_{\tilde{q}-\tilde{q}'}\Big][\hat{G}^\alpha]_{\tilde{q}'\tilde{q}''} = \delta_{\tilde{q},\tilde{q}''}. \tag{11}$$

Here $[G_0^\alpha(\tilde{q})]^{-1} = \mathrm{i}\tilde{\omega}_n - \boldsymbol{v}^\alpha \cdot \boldsymbol{q}$, where $\tilde{q} = [\boldsymbol{q}, \mathrm{i}\tilde{\omega}_n]$ involves fermionic Matsubara frequencies. Defining $V^\alpha(\boldsymbol{r},\tau) = \sum_q \mathrm{e}^{\mathrm{i}(\boldsymbol{q}\cdot\boldsymbol{r}-\omega_m\tau)}V_q^\alpha$ and

$$\mathcal{G}^\alpha(\boldsymbol{r},\boldsymbol{r}',\tau,\tau') = \frac{1}{\beta V}\sum_{\tilde{q}\tilde{q}'}\mathrm{e}^{\mathrm{i}(\boldsymbol{q}\cdot\boldsymbol{r}-\tilde{\omega}_n\tau)}\mathrm{e}^{-\mathrm{i}(\boldsymbol{q}'\cdot\boldsymbol{r}'-\tilde{\omega}_{n'}\tau')}[\hat{G}^\alpha]_{\tilde{q}\tilde{q}'} \tag{12}$$

it is easy to see that equation (11) is equivalent to

$$\big[-\partial_\tau + \mathrm{i}\boldsymbol{v}^\alpha\cdot\nabla_{\boldsymbol{r}} - V^\alpha(\boldsymbol{r},\tau)\big]\mathcal{G}^\alpha(\boldsymbol{r},\boldsymbol{r}',\tau,\tau') = \delta(\boldsymbol{r}-\boldsymbol{r}')\delta^*(\tau-\tau') \tag{13}$$

where $\delta^*(\tau-\tau') = \frac{1}{\beta}\sum_n \mathrm{e}^{-\mathrm{i}\tilde{\omega}_n(\tau-\tau')}$. Equation (13) can be solved exactly by means of a trivial generalization of Schwinger's ansatz [12]. The result is [6–8]

$$\mathcal{G}^\alpha(\boldsymbol{r},\boldsymbol{r}',\tau,\tau') = G_0^\alpha(\boldsymbol{r}-\boldsymbol{r}',\tau-\tau')\mathrm{e}^{\Phi^\alpha(\boldsymbol{r},\tau)-\Phi^\alpha(\boldsymbol{r}',\tau')} \tag{14}$$

$$G_0^\alpha(\boldsymbol{r},\tau) = \frac{1}{\beta V}\sum_{\tilde{q}}\frac{\mathrm{e}^{\mathrm{i}(\boldsymbol{q}\cdot\boldsymbol{r}-\tilde{\omega}_n\tau)}}{\mathrm{i}\tilde{\omega}_n - \boldsymbol{v}^\alpha\cdot\boldsymbol{q}} \tag{15}$$

$$\Phi^\alpha(\boldsymbol{r},\tau) = \sum_q \frac{\mathrm{e}^{\mathrm{i}(\boldsymbol{q}\cdot\boldsymbol{r}-\omega_m\tau)}}{\mathrm{i}\omega_m - \boldsymbol{v}^\alpha\cdot\boldsymbol{q}}V_q^\alpha. \tag{16}$$

Gaussian averaging of equation (14) with the effective action $S_{eff}\{\phi^\alpha\}$ given in equations (7) and (9) yields

$$\big\langle\mathcal{G}^\alpha(\boldsymbol{r},\boldsymbol{r}',\tau,\tau')\big\rangle_{S_{eff}} = G_0^\alpha(\boldsymbol{r}-\boldsymbol{r}',\tau-\tau')\mathrm{e}^{\mathcal{Q}^\alpha_{int}(\boldsymbol{r}-\boldsymbol{r}',\tau-\tau')} \tag{17}$$

where for patch-independent bare interactions of the form $[\underline{\tilde{f}}_q]^{\alpha\alpha'} = \frac{\beta}{V}f_q$ the Debye–Waller factor is given by

$$\mathcal{Q}^\alpha_{int}(\boldsymbol{r},\tau) = \frac{1}{\beta V}\sum_q \frac{f_q^{RPA}\,[1-\cos(\boldsymbol{q}\cdot\boldsymbol{r}-\omega_m\tau)]}{(\mathrm{i}\omega_m - \boldsymbol{v}^\alpha\cdot\boldsymbol{q})^2}. \tag{18}$$

Here $f_q^{RPA} = f_q[1+\Pi_0(q)f_q]^{-1}$ is the random-phase approximation for the effective interaction, and $\Pi_0(q) = \sum_{\alpha\alpha'}\Pi^{\alpha\alpha'}(q)$ is the total non-interacting polarization.

In the field theory literature [5] the auxiliary field ϕ^α is called *disorder field*. This terminology suggests that the functional integral formulation of bosonization can also be used to calculate the average Green's function of non-interacting electrons in a dynamic random potential. This connection between bosonization and disordered systems is obscured

in the operator approach [13, 14], but it is quite clear in the functional formulation. Indeed, in this work we shall show that the functional bosonization approach developed in [6–8] provides a useful method to calculate the average single-particle Green's function of electrons in a stochastic medium.

Let us first consider non-interacting spinless electrons at high densities subject to an *imaginary-time* random potential $U(\boldsymbol{r},\tau)$. The imaginary-time Green's function $\mathcal{G}(\boldsymbol{r},\boldsymbol{r}',\tau,\tau')$ is defined as usual

$$\left[-\partial_\tau - \frac{(-\mathrm{i}\nabla_{\boldsymbol{r}})^2}{2m} + \mu - U(\boldsymbol{r},\tau)\right]\mathcal{G}(\boldsymbol{r},\boldsymbol{r}',\tau,\tau') = \delta(\boldsymbol{r}-\boldsymbol{r}')\delta^*(\tau-\tau') \quad (19)$$

where m is the mass of the electrons. We assume that the random potential has a Gaussian probability distribution with zero average and general covariance $\overline{U(\boldsymbol{r},\tau)U(\boldsymbol{r}',\tau')} = C(\boldsymbol{r}-\boldsymbol{r}',\tau-\tau')$, where the overbar denotes averaging with respect to the probability distribution of the random potential. In Fourier space we have then $\overline{U_q U_{-q}} = \beta V C_q$, where the Fourier components are defined by

$$C_q = \int_0^\beta \mathrm{d}\tau \int \mathrm{d}\boldsymbol{r}\, \mathrm{e}^{-\mathrm{i}(\boldsymbol{q}\cdot\boldsymbol{r}-\omega_m\tau)} C(\boldsymbol{r},\tau) \quad (20)$$

and analogously for U_q. As pointed out in [2], the complicated quantum dynamics of an interacting many-body system can sometimes be described by an effective stochastic model. In this case the real-time quantum dynamics can be obtained from the imaginary time dynamics via analytic continuation. Moreover, imaginary time random potentials play an important role in the so-called directed polymer problem [3].

We are interested in the average Green's function $G(\boldsymbol{r}-\boldsymbol{r}',\tau-\tau') = \overline{\mathcal{G}(\boldsymbol{r},\boldsymbol{r}',\tau,\tau')}$. Within our bosonization approach $G(\boldsymbol{r}-\boldsymbol{r}',\tau-\tau')$ is calculated in the most direct way: first we obtain the exact Green's function $\mathcal{G}(\boldsymbol{r},\boldsymbol{r}',\tau,\tau')$ for a given realization of the random potential and then this expression is averaged. However, because the approximations inherent in higher-dimensional bosonization are only accurate for small momentum transfers, we need to assume that there exists a cut-off $q_c \ll k_F$ such that U_q becomes negligibly small for $|\boldsymbol{q}| \gtrsim q_c$. But this means that for wavevectors $|\boldsymbol{q}| \gtrsim q_c$ the Fourier coefficients C_q of the covariance function can be neglected. In other words, we can only study random potentials with sufficiently long-range spatial correlations, such that $q_c \ll k_F$. Evidently the most popular model of δ-function correlated disorder cannot be treated within our bosonization approach. However, in view of the fact that a random potential with a finite correlation range q_c^{-1} is expected to lead, for distances $|\boldsymbol{r}| \gg q_c^{-1}$, to results for single-particle properties qualitatively identical to those that would be obtained for a δ-function correlated random potential, this restriction does not seem to be very serious. By choosing the above-mentioned box cut-offs such that $q_c \ll \Lambda$, $\lambda \ll k_F$, we obtain cut-off-independent results for physical correlation functions at distances $|\boldsymbol{r}| \gg q_c^{-1}$.

As discussed above, due to the subdivision of the Fermi surface into small patches it is possible to linearize the energy dispersion locally within a given patch, $(\boldsymbol{k}^\alpha+\boldsymbol{q})^2/(2m)-\mu \approx \boldsymbol{v}^\alpha\cdot\boldsymbol{q}$, where $\boldsymbol{k}^\alpha$ is the centre of patch $\tilde{K}^\alpha_\Lambda$. We then replace equation (19) by a *first-order* partial differential equation for the patch Green's function $\mathcal{G}^\alpha(\boldsymbol{r},\boldsymbol{r}',\tau,\tau')$

$$\left[-\partial_\tau + \mathrm{i}\boldsymbol{v}^\alpha\cdot\nabla_{\boldsymbol{r}} - U(\boldsymbol{r},\tau)\right]\mathcal{G}^\alpha(\boldsymbol{r},\boldsymbol{r}',\tau,\tau') = \delta(\boldsymbol{r}-\boldsymbol{r}')\delta^*(\tau-\tau'). \quad (21)$$

The similarity between equations (21) and (13) is obvious. The solution of equation (21) can be read off from equations (14)–(16)

$$\mathcal{G}^\alpha(\boldsymbol{r},\boldsymbol{r}',\tau,\tau') = G_0^\alpha(\boldsymbol{r}-\boldsymbol{r}',\tau-\tau')\,\exp\left[\frac{1}{\beta V}\sum_q U_q \frac{\mathrm{e}^{\mathrm{i}[\boldsymbol{q}\cdot\boldsymbol{r}-\omega_m\tau]} - \mathrm{e}^{\mathrm{i}[\boldsymbol{q}\cdot\boldsymbol{r}'-\omega_m\tau']}}{\mathrm{i}\omega_m - \boldsymbol{v}^\alpha\cdot\boldsymbol{q}}\right]. \quad (22)$$

The Gaussian average of equation (22) yields the usual Debye–Waller factor

$$\overline{\mathcal{G}^\alpha(\boldsymbol{r},\boldsymbol{r}',\tau,\tau')} \equiv G^\alpha(\boldsymbol{r}-\boldsymbol{r}',\tau-\tau') = G_0^\alpha(\boldsymbol{r}-\boldsymbol{r}',\tau-\tau')e^{Q^\alpha_{dis}(r-r',\tau-\tau')} \tag{23}$$

with

$$Q^\alpha_{dis}(\boldsymbol{r},\tau) = -\frac{1}{\beta V}\sum_q C_q \frac{1-\cos(\boldsymbol{q}\cdot\boldsymbol{r}-\omega_m\tau)}{(\mathrm{i}\omega_m - \boldsymbol{v}^\alpha\cdot\boldsymbol{q})^2}. \tag{24}$$

The total disorder-averaged Matsubara Green's function can then be written as

$$G(\boldsymbol{k},\mathrm{i}\tilde{\omega}_n) = \sum_\alpha \Theta^\alpha(\boldsymbol{k})\int \mathrm{d}\boldsymbol{r}\int_0^\beta \mathrm{d}\tau\, \mathrm{e}^{-\mathrm{i}[(\boldsymbol{k}-\boldsymbol{k}^\alpha)\cdot\boldsymbol{r}-\tilde{\omega}_n\tau]}G_0^\alpha(\boldsymbol{r},\tau)\mathrm{e}^{Q^\alpha_{dis}(r,\tau)}. \tag{25}$$

Note that from equation (15)

$$G_0^\alpha(\boldsymbol{r},\tau) = \delta^{(d-1)}(\boldsymbol{r}^\alpha_\perp)\left(\frac{-\mathrm{i}}{2\pi}\right)\frac{1}{r^\alpha + \mathrm{i}|\boldsymbol{v}^\alpha|\tau} \tag{26}$$

where $\delta^{(d-1)}(\boldsymbol{r}^\alpha_\perp)$ is a Dirac δ-function of the $d-1$ components $\boldsymbol{r}^\alpha_\perp$ of $\boldsymbol{r}$ that are orthogonal to $\boldsymbol{v}^\alpha$. Because $G_0^\alpha(\boldsymbol{r},\tau)$ is proportional to $\delta^{d-1}(\boldsymbol{r}^\alpha_\perp)$, we may replace $Q^\alpha_{dis}(\boldsymbol{r},\tau) \to Q^\alpha_{dis}(r^\alpha\hat{\boldsymbol{v}}^\alpha,\tau)$ in equation (25), where $r^\alpha = \hat{\boldsymbol{v}}^\alpha\cdot\boldsymbol{r}$, with $\hat{\boldsymbol{v}}^\alpha = \boldsymbol{v}^\alpha/|\boldsymbol{v}^\alpha|$. This completes the solution of the non-interacting problem.

Because disorder and interactions are treated on an equal footing in our bosonization approach, it is easy to include the effect of electron–electron interactions. The random potential gives rise to an additional term

$$S_{dis}\{\psi\} = \beta\sum_q\sum_\alpha U_{-q}\rho^\alpha_q \tag{27}$$

in equation (2). This leads to the replacement $V^\alpha(\boldsymbol{r},\tau) \to (\mathrm{i}/\beta)\phi^\alpha(\boldsymbol{r},\tau) + U(\boldsymbol{r},\tau)$ in equation (13). To perform the averaging operation it is convenient to integrate *first* over the ϕ^α field and then over the disorder potential, because in this way we avoid the appearance of disorder-dependent denominators. Hence, it is *not* necessary to resort to the replica approach to perform the disorder averaging. As a result we find that for disordered interacting electrons equation (25) should be modified by replacing $Q^\alpha_{dis}(\boldsymbol{r},\tau)$ by

$$Q^\alpha_{tot}(\boldsymbol{r},\tau) = \tilde{Q}^\alpha_{dis}(\boldsymbol{r},\tau) + Q^\alpha_{int}(\boldsymbol{r},\tau) \tag{28}$$

with $Q^\alpha_{int}(\boldsymbol{r},\tau)$ given in equation (18). The contribution $\tilde{Q}^\alpha_{dis}(\boldsymbol{r},\tau)$ in equation (28) is formally identical with $Q^\alpha_{dis}(\boldsymbol{r},\tau)$ in equation (24), except that the bare covariance function C_q should be replaced by the screened covariance function

$$\tilde{C}_q = \frac{C_q}{[1+\Pi_0(q)f_q]^2}. \tag{29}$$

The denominator in this expression has a simple physical interpretation: it describes the screening of the disorder potential by the electron–electron interactions. In $d = 1$ a result similar to equation (28) has recently been obtained by Kleinert [15], and by Hu and Das Sarma [16]. While the latter authors did not obtain the screening of the impurity potential, the expression given by Kleinert [15] is exactly recovered from our result by setting $d = 1$. Note, however, that Kleinert has combined functional bosonization [4] with the replica approach to handle the disorder averaging. As already mentioned, we can avoid the introduction of replicas by integrating *first* over the Hubbard–Stratonovich field and average over the disorder at the end.

Evidently equation (28) does not contain interference terms describing weak localization effects. These are known to play an important role in the low-energy behaviour of the

average Green's function of an interacting disordered Fermi system in the metallic regime. Note that diagrammatically weak localization is described in terms of infinite impurity ladders called Cooperon and Diffuson propagators [17], which satisfy the diffusion equation. But the diffusive motion involves large changes in the direction of the particle due to many successive (possibly small-angle) scatterings. Such a motion cannot be correctly described within the approximations inherent in higher-dimensional bosonization at the level of the Gaussian approximation [6, 7, 13, 14] (which amount to (i) the neglect of momentum transfer between different patches and (ii) the local linearization of the energy dispersion [8]), because in this case the electron trajectories are approximated by straight lines in the directions of the local velocities $\boldsymbol{v}^\alpha$. Obviously the weak localization effects must be contained in the corrections to the straight-line approximation for the electron trajectory. Hence, the physics of weak localization will only emerge if we can generalize our approach such that it describes changes in the direction of the electron propagation due to successive scatterings. This can be achieved *either* by taking momentum transfer between different patches (so-called around-the-corner processes) into account, *or* by considering patches with a finite curvature (i.e. by retaining the quadratic terms in the expansion of the energy dispersion at the Fermi surface) [18]. The equivalence of these procedures follows from the fact that any patch with a finite curvature can be subdivided into a larger number of approximately flat sub-patches, such that scattering from the original curved patch can also be described in terms of scattering from the set of *coupled* but flat sub-patches. The latter point of view leads, in the calculation of the Green's function for fixed background field [8], to a system of coupled first-order differential equations, which can always be transformed into a smaller system of differential equations with higher-order derivatives. A systematic method for including the non-linearities in the energy dispersion into higher-dimensional bosonization has recently been developed in [19]. At present it is not clear whether it is possible to obtain weak localization effects with this approach.

Let us now show that equation (25) reduces for static disorder to the usual perturbative result. In this case only the $\omega_m = 0$ component of the covariance function C_q is non-zero. For simplicity let us assume that C_q has a separable form, $C_q = \delta_{\omega_m,0}\beta\gamma_0 \mathrm{e}^{-|q|_1/q_c}$, where $|\boldsymbol{q}|_1 = \sum_{i=1}^d |q_i|$ [20]. Then we obtain from equation (24) for $V \to \infty$

$$Q^\alpha_{dis}(r^\alpha \hat{\boldsymbol{v}}^\alpha, \tau) \sim -\frac{|r^\alpha|}{2\ell^\alpha} \qquad |r^\alpha q_c| \gg 1 \tag{30}$$

where the inverse elastic mean free path is given by $\frac{1}{\ell^\alpha} = \frac{q_c}{\pi}^{d-1} \frac{\gamma_0}{|\boldsymbol{v}^\alpha|^2}$. We conclude that at large distances $G^\alpha(\boldsymbol{r}, \tau) = G^\alpha_0(\boldsymbol{r}, \tau)\mathrm{e}^{-|r^\alpha|/2\ell^\alpha}$. In Fourier space this implies for $|\boldsymbol{q}| \ll q_c$

$$G(\boldsymbol{k}^\alpha + \boldsymbol{q}, \mathrm{i}\tilde{\omega}_n) = \frac{1}{\mathrm{i}\tilde{\omega}_n - \boldsymbol{v}^\alpha \cdot \boldsymbol{q} + \mathrm{sign}(\tilde{\omega}_n)(\mathrm{i}/2\tau^\alpha)} \tag{31}$$

where $1/\tau^\alpha = |\boldsymbol{v}^\alpha|/\ell^\alpha$ is the inverse elastic lifetime. It is easy to see that this result agrees with the usual perturbative expression for the average self-energy in the lowest-order Born approximation, which is given by [21]

$$\mathrm{Im}\Sigma(\boldsymbol{k}) = \frac{\gamma_0}{V} \sum_{\boldsymbol{q}} \mathrm{e}^{-|q|_1/q_c} \mathrm{Im}G_0(\boldsymbol{k} + \boldsymbol{q}, -\mathrm{i}0^+). \tag{32}$$

Because for $q_c \ll k_F$ only wavevectors with $|\boldsymbol{q}| \ll k_F$ contribute, we are allowed to approximate the Green's function on the right-hand side of equation (32) by its linearized form. Then we obtain

$$\frac{1}{\tau^\alpha} = 2\mathrm{Im}\Sigma(\boldsymbol{k}^\alpha) = \left(\frac{q_c}{\pi}\right)^{d-1} \frac{\gamma_0}{|\boldsymbol{v}^\alpha|} \tag{33}$$

in agreement with the above bosonization result. In [22] we have shown that a Debye–Waller factor $Q^{\alpha}(F,T)$ which diverges for $r=0$ and $|r^{\alpha}|\to\infty$ as a power law ($\sim|r^{\alpha}|^{\mu},\ \mu>0$) completely washes out any singularity in the momentum distribution $n_{\boldsymbol{k}}$. Hence the average momentum distribution is for any finite disorder analytic at the Fermi surface. Of course, this is a well known result [21]. It is also easy to understand why in one-dimensional interacting Fermi systems any finite disorder destroys the algebraic singularity of $n_{\boldsymbol{k}}$, which is one of the characteristics of a Luttinger liquid [23]. Recall that this algebraic singularity is due to the logarithmic divergence of $Q^{\alpha}_{int}(\boldsymbol{r},0)$ for $r^{\alpha}\to\infty$. At sufficiently large distances this weak logarithmic divergence is negligible compared with the linear divergence of $Q^{\alpha}_{dis}(\boldsymbol{r},0)$.

The case of a time-dependent random potential is more interesting. To calculate the average Green's function, we should specify the dynamic covariance function C_q. If we would like to model an underlying interacting many-body system in thermal equilibrium by a random system, then the form of C_q is completely determined by the nature of the interaction. In the case of the coupled electron–phonon system at high temperatures an explicit microscopic calculation of C_q has been given by Girvin and Mahan [2]. However, their identification of C_q with the parameters of the underlying many-body system is based on a perturbative calculation of the self-energy at high temperatures. *Our functional bosonization approach allows us to relate the covariance function C_q of the random system in a much more direct and essentially non-perturbative way to the underlying many-body system.* Evidently, the requirement that the average Green's function of the stochastic model should be identical with the Green's function of the many-body system without disorder is equivalent to $Q^{\alpha}_{dis}(\boldsymbol{r},\tau)=Q^{\alpha}_{int}(\boldsymbol{r},\tau)$, where $Q^{\alpha}_{dis}(\boldsymbol{r},\tau)$ and $Q^{\alpha}_{int}(\boldsymbol{r},\tau)$ are given by equations (24) and (18). It immediately follows that the connection between the effective stochastic model and the interacting many-body system is given by the surprisingly simple relation

$$C_q=-f_q^{RPA}. \tag{34}$$

For example, to describe longitudinal acoustic phonons with dispersion $\omega_{\boldsymbol{q}}$ that are coupled to the electrons via the Coulomb potential $f^{cb}_{\boldsymbol{q}}=4\pi e^2/\boldsymbol{q}^2$, we should choose the covariance function [24]

$$C_q=-\frac{f^{cb}_{\boldsymbol{q}}}{1+f^{cb}_{\boldsymbol{q}}\Pi_{ph}(q)}\qquad \Pi_{ph}(q)=\Pi_0(q)+\frac{\gamma\omega^2_{\boldsymbol{q}}}{\omega_m^2+\omega^2_{\boldsymbol{q}}} \tag{35}$$

where γ measures the strength of the electron–phonon coupling. We would like to emphasize that in spite of its apparent simplicity, equation (34) is a highly non-trivial result, because it is based on a controlled summation of the entire perturbation series of the many-body problem via bosonization. The crucial point is that *bosonization produces an exponential resummation of the perturbation series, so that the effect of the interactions on the Green's function can be expressed exclusively in terms of a Debye–Waller factor $Q^{\alpha}_{int}(\boldsymbol{r},\tau)$, which can then be directly compared with the Debye–Waller factor $Q^{\alpha}_{dis}(\boldsymbol{r},\tau)$ due to disorder.*

Of course, the dynamic random potential could also be due to some non-equilibrium external forces, in which case the above identification with an underlying many-body system is meaningless. As an example, let us consider a Gaussian white-noise random potential, with covariance given by $C_q=C_0\mathrm{e}^{-|\boldsymbol{q}|_1/q_c}$. Substituting this into equation (24) and taking the limit $V,\beta\to\infty$, the integrations can be performed analytically, with the result

$$Q^{\alpha}_{dis}(r^{\alpha}\hat{\boldsymbol{v}}^{\alpha},\tau)=\frac{\mathrm{i}W\tau}{r^{\alpha}+\mathrm{i}|\boldsymbol{v}^{\alpha}|\tau+\mathrm{i}\,\mathrm{sign}(\tau)q_c^{-1}} \tag{36}$$

where $W=(C_0/2\pi)(q_c/\pi)^{d-1}$. Because the Debye–Waller factor vanishes at $\tau=0$, we have $G^{\alpha}(\boldsymbol{r},0)=G^{\alpha}_0(\boldsymbol{r},0)$, so that the dynamic white-noise random potential does not affect

the momentum distribution n_k. It is easy to see that this is an artefact of the white-noise limit. Note, however, that for $\tau/r^\alpha \to \infty$

$$G^\alpha(\boldsymbol{r},\tau) \sim -\delta^{(d-1)}(\boldsymbol{r}_\perp^\alpha)\frac{\mathrm{e}^{W/|\boldsymbol{v}^\alpha|}}{2\pi|\boldsymbol{v}^\alpha|\tau}. \tag{37}$$

The above limit determines the density of states at the Fermi energy, which in turn can be expressed in terms of a renormalized effective mass. Hence, the random potential enhances the effective mass by a factor of $\mathrm{e}^{W/|\boldsymbol{v}^\alpha|}$. This is intuitively clear: the electrons become heavier because they have to overcome the resistance of the random potential. The Fourier transformation of $G^\alpha(\boldsymbol{r},\tau)$ can be calculated exactly. For $\hat{\boldsymbol{v}}^\alpha \cdot \boldsymbol{q} \geqslant 0$ the result is

$$G^\alpha(\boldsymbol{q},\mathrm{i}\omega) = \frac{1}{Wq_c + \mathrm{i}\omega - \boldsymbol{v}^\alpha\cdot\boldsymbol{q}}\left\{1 + \frac{Wq_c\mathrm{e}^{-(\hat{\boldsymbol{v}}^\alpha\cdot\boldsymbol{q})/q_c}}{\mathrm{i}\omega - \boldsymbol{v}^\alpha\cdot\boldsymbol{q}}\exp\left[-W\frac{\hat{\boldsymbol{v}}^\alpha\cdot\boldsymbol{q}}{\mathrm{i}\omega - \boldsymbol{v}^\alpha\cdot\boldsymbol{q}}\right]\right\}. \tag{38}$$

If we now analytically continue this expression to real frequencies by replacing $\mathrm{i}\omega \to \omega + \mathrm{i}0^+$, we encounter an essential singularity at $\omega = \boldsymbol{v}^\alpha\cdot\boldsymbol{q}$. We suspect that this is an artefact of the linearization.

In summary, we have shown that functional bosonization can be used to calculate the disorder-averaged Green's function of electrons at high densities that are subject to a random potential with long-range spatial correlations. While in the static limit we have recovered the usual perturbative result, for time-dependent random potentials we have obtained a highly non-trivial expression for the averaged single-particle Green's function. One of our main results is equation (34), which puts the description of an interacting many-body system via an effective stochastic model on a solid microscopic basis. Although our method describes disorder and interactions on an equal footing, the corresponding contributions to the Debye–Waller factors are (apart from the screening of the random potential) additive, so that interference terms containing weak localization effects do not appear. However, it might be possible to calculate the single-particle Green's function via higher-dimensional bosonization beyond the Gaussian approximation, taking momentum-transfer between different patches or the non-linearities in the energy dispersion approximately into account [7, 19]. In this case bosonization in $d > 1$ could become a new powerful approach to disordered interacting Fermi systems in the diffusive regime, which can deal simultaneously with disorder and interactions and does not have the disadvantages of the replica trick. Note that the diffusive regime does not exist in $d = 1$, so that the physics of weak localization can only be discussed within higher-dimensional bosonization. Work in this direction is in progress.

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